

Inversion of Equiangular Spirals

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Abstract

In this article we focus on properties of equiangular spirals and their inversion in a circle. Our aim is to emphasize the use of polar coordinates and describe how to draw the inversion of equiangular spirals to get two-pole spirals. In particular, we provide a connection to the art of Escher through the use of GeoGebra software. This allows to create exploring activities that enrich the teaching process and expose both students and instructors to the beauty of mathematics.

1 Introduction

A *spiral* is a curve in the plane that winds continuously about a fixed point called a *pole*. *Equiangular* spirals or the *Logarithmic* spirals, are plane curves with polar equation $r = e^{a\theta+b}$, where a and b are real numbers. The main property of equiangular spirals is that the tangent at any point on the curve makes a constant angle with the radial line that issues from the pole and passes through the point. The spiral of Archimedes with polar equation $r = a\theta$, and Fermat's spiral with polar equation $r = a\sqrt{\theta}$, where $a \neq 0$ is a fixed real number, are not equiangular. To see this for the spiral of Archimedes, we draw with GeoGebra the polar equation of the spiral of Archimedes for the value $a = 1$. In the input bar of GeoGebra we insert the equation $r = \theta$, the software will recognize that we are working with polar coordinates. The obtained curve is shown in Fig. 1. We mark two points B and C and draw the radial lines, using the ray tool, between the pole A and each one of the points B and C . Then using the tangent tool we draw the tangents of the curve at the points B and C . Finally, we use the angle tool to measure the angle between these tangents and the radial lines. We see that the angle is not preserved, see Fig. 1.

On the other hand, the spiral $r = e^\theta$ is equiangular as shown in Fig. 2, the tangent at any point on the curve makes a constant angle with the radial lines, see also the animated Figure [10].

*This topic was presented at CADGME 2022 conference

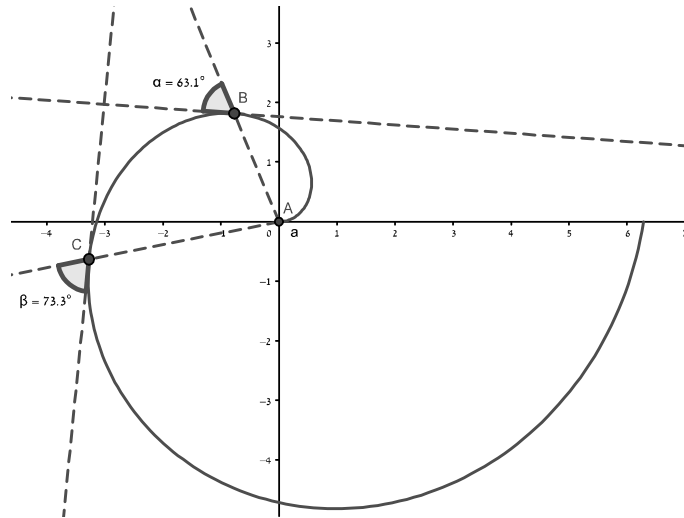


Figure 1: The spiral of Archimedes with polar equation $r = \theta$ is not equiangular

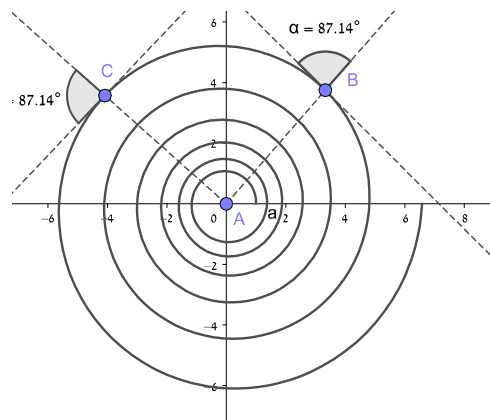


Figure 2: Equiangular spiral

The equiangular spiral was first considered in 1638 by Descartes. His discussion was based upon the consideration of a curve cutting radii vectors from a fixed point O under a constant angle, ϕ . From this he inferred the polar equation $r = ke^{c\theta}$ where $c = \cot \phi$. When $c = 1$, then the ratio of two radii vectors corresponds to a number and the angle between them to its logarithm; whence the name of the curve, see [1].

Torricelli, who died in 1647, worked on it independently and found the length of the curve. He used for a definition the fact that the radii are in geometric progression if the angles increase uniformly, i.e., for each angle θ_0 the radial line $\theta = \theta_0$ meets the spiral at infinitely many discrete points. The radii vectors from the center of the spiral to these points form a geometric progression.

Jacob Bernoulli, some fifty years later, found all the “reproductive” properties of the curve; and these almost mystic properties of the “wonderful” spiral made him wish to have the curve incised on his tomb: *Eadem mutata resurgo: “Though changed I rise unchanged”*, see [1] and [6].

In this article, we focus on inversion of equiangular spirals in a circle. We bring a discussion of Coxeter on the whirlpool of Escher, and through the use of GeoGebra software we experiment in drawing two-pole spirals. Teachers and instructors at schools and colleges often use the Cartesian coordinates in the graphic window of GeoGebra. Our aim is to emphasize the use of polar coordinates with equiangular spiral curves. This allows to create exploring activities that may enrich the teaching process and expose both students and instructors to the beauty of mathematics.

2 Inversion of equiangular spirals

Inversion in a circle with center O and radius R is a transformation of the plane that maps each point P to a radial point P' (a point that lies on the line issuing from O) such that $OP \cdot OP' = R^2$. In reference to the unit circle, inversion is defined in the extended complex plane by rule:

$$I : C^* \rightarrow C^*$$
$$I(z) = \frac{1}{\bar{z}}.$$

Indeed, let $z = re^{i\theta}$ be any point in extended complex plane, then $\frac{1}{\bar{z}} = \frac{1}{r}e^{i\theta}$ lies on the same ray issuing from the origin through z and $|z| \cdot \left|\frac{1}{\bar{z}}\right| = \frac{1}{r} \cdot r = 1$. Therefore, in polar coordinates (r, θ) inversion in the unit circle is given by

$$(r, \theta) \rightarrow \left(\frac{1}{r}, \theta\right).$$

Inversion in a circle reverses direction: if the point approaches infinity then its inverse approaches the center of the circle and if the point winds in clockwise

direction around the pole on the curve then its inversion winds in anti clockwise direction around the image of the pole.

Points on the circle of inversion are preserved and points inside the circle are mapped one-to-one to points outside the circle and vice versa. In particular, inversions map equiangular spirals to equiangular spirals. To see this fact using GeoGebra, we apply the following steps: In fact, GeoGebra provides inversion by using a simple tool “reflection about a circle”.

- Draw the unit circle with center $O(0, 0)$ and radius 1.
- In the input bar insert the polar equation $r = e^{(\frac{0.7\theta}{2\pi} + 0.1)}$ of an equiangular spiral.
- In the input bar insert the equation of the spiral $r = e^{(-\frac{0.7\theta}{2\pi} - 0.1)}$.
- By drawing the tangent lines, measuring the angles between the radial lines and the tangents and dragging points A and D , we can validate that both curves are equiangular (see Fig. 3).

Measuring angles with GeoGebra is done by using the angle tool and clicking on the two rays that define the angle. In Fig. 3, when animation is on point A along the original spiral we see that the angle, the radial lines and the tangents is preserved. Likewise, when animation is on point D along the inversion spiral the angle the radial lines and the tangents is preserved, see the animated Figure [11].

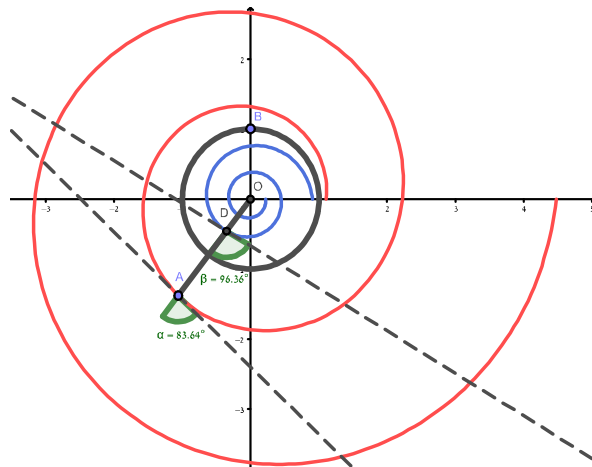


Figure 3: Both spiral and its inversion are equiangular

Another important feature of equiangular spirals and their inversion is that when θ increases arithmetically then r will increase geometrically. We can check this property in GeoGebra, by drawing one radial line and find the intersection points with the spiral or its inversion. Measuring the distance of these points from O yields a geometric progression. Indeed, if (r_1, θ_1) and $(r_2, \theta_1 + 2\pi)$ are

two consecutive intersection points of the radial line with the spiral $r = e^{a\theta+b}$ then $r_1 = e^{a\theta_1+b}$ and $r_2 = e^{a(\theta_1+2\pi)+b} = e^{2\pi a}r_1$.

2.1 Two-pole spirals

Two-pole spirals can be obtained by inverting a spiral, the pole of which lies outside the inversion circle. In fact, a spiral has one finite pole in the plane and a second pole at infinity. Inversion in this case creates a spiral inside the circle with two poles: one pole at the center of the inversion circle and another pole at the image of the pole in the original curve.

The following procedure illustrates how to get two-pole spirals with GeoGebra.

1. In the input bar insert the equation of the family of equiangular spirals $r = e^{\frac{a}{2\pi}\theta+b}$.
2. Define sliders for a and b , $0 < a < 1$ with increment 0.01 and $0 \leq b \leq 1$ with increment 0.1.
3. Enlarge the domain of θ to 20π .
4. Draw a circle with center A and radius AB such that the circle does not contain the pole nor intersect the curve as shown in Fig. 4.
5. Mark a tracing point C on the curve and reflect C about the circle with center A and radius AB .
7. Finally, trace the inversion point C' of C and turn animation on for point C .

Fig. 4, shows a two-pole spiral which is the image by an inversion of the curve $r = e^{\frac{a}{2\pi}\theta+b}$ for the values $a = 0.68$ and $b = 1$, see also the animated Figure [12] with other values for a and b .

2.2 Connection to the art of Escher

Coxeter discussed Escher's *whirlpool* (see [8] (minute 41), and [9]). He first explains what is inversion in a circle:

“One way to bring infinity to finite terms is by the transformation called inversion in a circle, whereby a point outside the circle is transformed into a point inside the circle so that it is on the same diameter but the distance from the center is reciprocal of the distance to the original point. In this way, if a point moves outside the circle, the inverse point moves inside and even if the point moves infinitely far away its image is still inside the circle. As the point moves farther and farther away, the image point gets closer and closer to the center”.

Then Coxeter describes the inversion of equiangular curve:

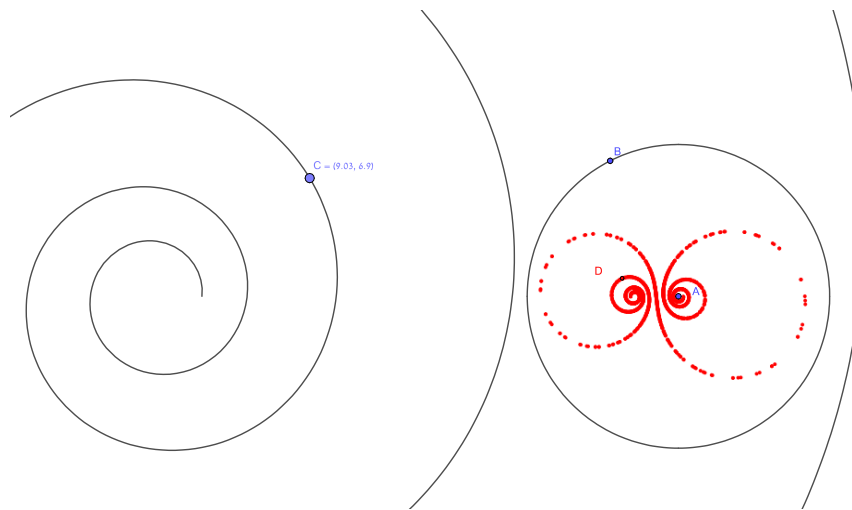


Figure 4: A two-pole spiral for the values $a = 0.68$ and $b = 1$

“If one considers the important curve called an equiangular spiral, which is a curve that goes round and round one point, that curve is inverted in a circle (Coxeter draws a circle between two rounds of the curve), we get a kind of spiral with one pole at the inverse of the original pole, going the opposite way round. But, the part that goes to infinity makes a second pole at the center. So, you have the curious effect of a curve that has two poles and that is the curve which Escher used in his work”.

This short discussion of Coxeter on Escher’s whirlpool is found also in the book [3, p. 290]. In fact, Coxeter asserts that Escher used equiangular spirals in this discussion, although in [2] he emphasizes that Escher’s work was based on his intuition rather than computations. So, the main question is why Escher used inversion of equiangular spirals? The answer could be hidden in the main feature of these spirals: If θ increases arithmetically then r will increase geometrically by a constant ratio q and the painter can easily resize his pattern by this constant ratio q .

In 1957 Escher finished his woodcut and wood engraving Whirlpools [9]. He used two inversions of equiangular spirals by drawing two curves on which the fish swim in opposite directions. In fact, Escher used a new printing technique for it, cutting one block which he printed on the same piece of paper in two colours. Two rows of fish swimming head to tail fill the space. The red row has exactly the same shape as the grey one, but has been turned 180 degrees. It seems that Escher painted a row of red fish swimming on the inversion of an equiangular spiral on a rectangular piece of paper and finished the wood engraving, then he turned around the piece of paper in 180 degrees and finished the wood engraving to get another row of grey fish swimming head to tail and

filling the space.

In GeoGebra, see Fig. 5, we can draw directly the inversions of two equiangular spirals: $r = e^{\frac{0.89\theta}{2\pi}+2}$ and $r = e^{\frac{0.83\theta}{2\pi}+1}$, $0 < \theta < 20\pi$. The reader is encouraged to change slightly the coefficients of θ to see how the view of the curves is resized inside the circle.

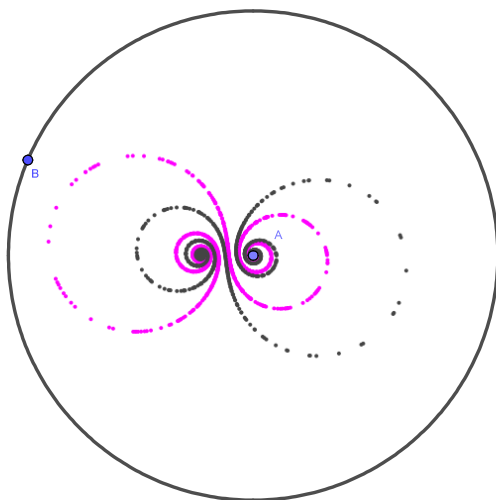


Figure 5: Inversions of two equiangular spirals

3 Concluding remarks

Many other researchers were fascinated with equiangular spirals. The authors of [7] used both hand-drawn and computer-drawn graphics with conformal mappings to generate advanced Escher-like spiral tessellations. In [1], the author reviews the history of discovering the logarithmic (equiangular) spirals and their properties between the seventeen to nineteen centuries. The author of [4], describes and illustrates two conformal mappings, inversion and anti-Mercator, and the symmetry implications of their combined use. The author of [5], investigated the logarithmic spirals in nature by means of dynamic geometry and computer algebra systems and discussed how we can combine the pupil's knowledge with the computational and graphical power to model some real-world phenomena.

The implication for teaching is that instructors can use GeoGebra to create exploring activities such as: drawing different kinds of spirals, checking whether a spiral is equiangular or not, drawing the inversion of a spiral, checking whether the inverted spiral is equiangular or not, and drawing two-pole spirals.

Finally, concerning the two-pole spirals, the use of the family of equiangular spirals $r = e^{\frac{a}{2\pi}\theta+b}$ with sliders for a and b in GeoGebra allows us to create

different kinds of inverted curves. We recommend the reader to try executing the inversion of these curves for different values of a and b and compare the result with other families of spirals not necessarily equiangular.

Acknowledgement. The author is indebted to the reviewers for their comments and valuable suggestions that improved the exposition of the paper. Special thanks are due to Csaba Sárvári for communicating and handling the manuscript.

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